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[SECOND SERIES.]

Art. X.—*Some New Discoveries respecting the Dates on the great Calendar Stone of the Ancient Mexicans, with Observations on the Mexican Cycle of Fifty-two Years;* by E. G. Squier, New York.

The most interesting monument of antiquity which has been discovered in America, is unquestionably the great Calendar Stone of the Aztecs, which now occupies a place in the walls of the Cathedral of the city of Mexico. It is an immense mass of porphyry, estimated to have weighed originally upwards of thirty tons. Its horizontal face is inscribed with a circle in relief, within which is found a complication of signs and figures, chiefly of an astronomical character, and referring to the motions of the sun. The relative positions and dependencies of these signs cannot be indicated without the aid of an engraving. I shall therefore, without going into a particular account of the stone,—involving, as it necessarily would, a complete analysis of the Aztec Calendar,—simply call attention to some of the results which have attended its study by Gama, Humboldt, Gallatin and others, so as to be able to submit, in a comprehensible manner, some additional discoveries which have followed its investigation, under more favorable circumstances.

The authorities above named, ascertained the existence of five signs upon this stone, referring to the principal annual positions of the sun, viz: the dates of the two transits of the sun by the zenith of Mexico, the dates of the vernal and autumnal equinoxes, and the date of the summer solstice. The summer solstice, according to the stone, occurred on the 22d of June; the sulphuric acid at present, but shall defer the consideration of the subject until the publication of my report on the mineral springs of Canada, which will be accompanied with the analyses of this water as collected in different years. Hoping that my observations may resolve a hitherto unexplained problem in the geology of this region, I beg leave to submit them to the notice of the Association.

ART. XV.—*On the Fundamental Principles of Mathematics;* by STEPHEN ALEXANDER, Professor of Mathematics and Astronomy in the College of New Jersey.

THE object of this dissertation is to present, to some extent at least, those ultimate principles and reasons, on which are founded the conclusions of mathematics; principles and reasons which lie beneath the artificial symbols which the science employs; i. e. —to borrow a most expressive and beautiful figure—an attempt will be made to seize upon and exhibit, in so far as may be, that "central thread of common sense, on which the pearls of analytical research are invariably strung."*

In pursuance of the object thus indicated, a definite arrangement will be made of the several topics to be specially considered, each under its own descriptive title; commencing with the following:

The Characteristics of Truth, especially Mathematical Truth.

(1.) *Truth*, which may be employed to designate the great object of all scientific research, is a term much too valuable to be misunderstood, but withal so general as not to admit of a ready definition. Yet, under its various aspects, truth will be found to present the characteristic feature of consistency with some great standard. Thus a careful consideration of its subjects of research will show that

That is *true* in *Mathematics*, which, under the existing system of things, is *supposable.* It is in no case requisite that the supposition should have been realized.

[No one, for instance, has ever seen a perfect circle; yet every one, who has carefully considered the matter, can clearly understand what a perfect circle ought to be, and will concede that its existence is entirely possible, perfectly *supposable.*†]

That is *true* in *Physics*, or (with some restrictions of application) in *Metaphysics*, which has been *permitted to exist.*

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* Sir J. F. W. Herschel—Cabinet Cyclopædia, Treatise on Astronomy, (10).

† Among the truths *supposable*, must even be classified those expressed by the aid of what are termed *imaginary quantities*.

[It *might be true*, in so far as we can discern, that heavy bodies should have been so constituted as in every case, to fall to the earth in an arc of a circle, or any other given curve. It *is a fact* that heavy bodies, falling freely, always fall to the earth in straight lines.

Glass might, perhaps, be so constituted under a new arrangement of things, that every known liquid would dissolve it; and this, without ceasing to be essentially what it now is. As things are now constituted, scarcely one or two of the known liquids will touch it.

In *Metaphysics*, moreover, one of the primary inquiries is not, whether it be *supposable* that man should be endowed with such faculties as reason, memory, &c.; but whether he *in fact* possesses them.]

Not to multiply examples, but recapitulating the characteristics already mentioned, we observe, that

That is *true* in *Mathematics*, which, under the existing system of things, is *supposable*—*That* is *true* in *Physics* and (with some restrictions) in *Metaphysics*, which has been *permitted to exist.*

That is *true*, in *matters of Taste*, which is *consistent with the laws of beauty* founded upon the *relations* of *things actual*—and *that* is true in *Morals* (in the highest and best sense; in which it is good);—*that* is *true*, in this sense, which is *consistent with what is to be found in the* GREAT SOURCE OF ALL GOOD. Or in general—due regard being had to its object—it may be asserted, that it is the *perfect consistency* with that which *may be*, or that which *is*, or that which *ought to be*, that constitutes the great characteristic feature of TRUTH.

Objects of Mathematical Research.

(2.) Mathematics, whether pure or mixed, has never to do with *things* as such, but only with the *relations of things.*

This is most manifestly true with regard to number, length, surface, capacity or volume, &c. For there are no such *things* as 2, 3, $7\frac{1}{2}$, &c., separately considered; nor can length, breadth, thickness, &c., exist apart from the things to which they belong, but only that *room* for them, which is to be found in space.

Thus also, mechanical force, motion or (in its qualified form) velocity, rest, time, &c., in so far as we have to do with them, exist not as *things*, but as the *relations of things actual;* and even the earth's orbit no where exists definitely in space; though, being a disturbed ellipse, of known dimensions, &c., it may be accurately prescribed.

(3.) Upon the fundamental fact, thus exhibited, depends the accuracy of mathematical reasoning. For, the relations of things with which it has to do, admit of being accurately ascertained and defined; which can by no means be always asserted of the inherent properties of the things themselves; i. e., those which render the things to which they belong what they are.

For while the form, size, &c., of a body can in many cases be accurately ascertained, and the supposable mathematical form which that of the body may closely resemble, may have perfectly definite and well ascertained properties, we, as yet, can know very little of the atoms which compose that body, and cannot even assert that, in the strict sense of the word, it is composed of atoms at all.

It cannot then be a legitimate objection to the conclusions of mathematics, that there *are no such things*, as those to which they refer; since those conclusions have not to do with things as such, but with their *relations*, and as stated at the outset, (1.) it is sufficient that even these be *supposable*, constituted as things now are.

The Relations of Things are Matters of Constitution and Arrangement.

(4.) The relations of things already designated are themselves not mere figments of the human mind, but—as all experience teaches us—they are *constituted* relations: i. e., in so far as we have to do with them, their connexion with things actual is a matter of arrangement dependent upon the constitution of the things, or else the things themselves are in some measure constituted in subordination to those relations: or both.

Thus, one part of space is not diverse from another, nor does one day of the week of course succeed another, because we may *choose to think so*, but because the CREATOR has formed (or *conformed*) them so. For "of" HIM not merely are all things, but "by" HIM they also *consist:* or, in other words, HE has not merely made those things with which we are familiarly conversant, *what* they are, but also, in certain respects, *as* they are. Any similarity in the relations of things must therefore also be a matter of constitution or arrangement; and we may safely make use of it in the illustration of one class of relations, by a comparison with another.

Of Quantity and its Distinctions and Ratio.

(5.) *Quantity* is the general term employed to designate all those relations of things which are the subjects of investigation in mathematics. In so far as it is thus employed, it denotes whatsoever admits of the distinction of greater and less.

(6.) Two quantities are of *the same species*, if each, *in itself*, exceeds its less, in the self same respect in which the other, *in itself*, exceeds its less; i. e., the terms greater and less must be applicable, in the case of each, in the self same sense. They must, moreover, be thus applicable to the quantities themselves, and not merely to their boundaries or limits.

Thus a straight line and a curve are of the same species; since each exceeds its less in length—in which respect alone a line can be either great or small. But, a straight line and a square are of different species; since the one exceeds its less in length, while the other exceeds its less in surface; and this, although the boundaries or limits of the square are, themselves, straight lines.

A straight line and an hour are quantities of a different species, since the one exceeds its less in length, but the other exceeds *its* less (e. g., a minute) in duration.

(7.) So fundamental and inherent is the distinction between quantities of different species, that the combination of them by addition, or the attempt to subtract one from the other, or to compare what constitutes greatness in the one species with that which constitutes it in the other, will all be found to be impracticable, and even manifestly absurd.

Thus a straight line cannot be added to a day, nor a pound in weight be subtracted from the surface of a triangle; nor can we say of an hour and a square that one is larger than the other, or even compare them at all as to greatness.

The single point of resemblance between quantities of different species, is that indicated (5.) in the definition already given of quantity in general; viz., that the distinction of greater and less *in some sense*, is every where admissible. Hence it is possible to compare the ratio of two quantities of one species with that of two other quantities of another species; or even that an equality of such ratios should exist; one of the first pair being precisely as great or small in comparison with the other, in the peculiar sense of great or small which belongs to that species, as one of the other pair of quantities is great or small in comparison with the other, though in the peculiar sense of great or small which belongs to *that* species. Thus, 2 feet : 1 foot : : 2 hours : 1 hour.

Of the Limits of Various Quantities, and the Nature of Zero.

(8.) The nature of the boundaries or limits of the quantities of various species will next be considered; and this will naturally lead to an examination of the nature of *zero*.

The limits of bounded space being the most obvious, and also those with which we are most familiar, may well claim our attention first.

Solids, or rather volumes, *occupy* space; and their limits are surfaces. In accordance with what has already been advanced, (2.) it will be observed, that it is with the form, capacity, &c., of the space thus occupied, that the mathematician, as such, has to do, and not with the nature of the substance to which they may appertain.

A surface (i. e., the very outside) of a solid, although it bounds that solid, is yet no part of the solid itself. To remove or even separately mark out any *portion* of the solid, the region at which the division is made or indicated, must lie beneath the surface. The surface exists only where the solid, or the space occupied by the solid, ends and other space begins. The surface itself occupies space not at all; it only *divides* space. It is not *somewhat*, in the same sense in which the solid is somewhat, but only *somewhere:* viz., where, as already stated, the solid ends, and space exterior to it begins.

The surface, then, having no capacity, is in that respect a *zero* of solidity; and we may with propriety say, when a solid such as a cube or a parallelopiped is reduced to its base (its altitude being reduced to *zero*), that the solid (as such) is reduced to *zero.*

The base or other surface, though thus a *zero* of capacity, is yet *somewhat* in its own sense—in the sense peculiar (6.) to that species of quantity—viz., *in superficial extent;* i. e., it still possesses, as it were, the property of covering or extending over, as well as limiting, a portion of the solid, and also that of *dividing* space.

But *a line* existing at the edge of such a surface, or any other, is not *somewhat*, even in the sense last mentioned, but only *somewhere:* viz., at the very edge of the surface. It does not *divide*, but only *penetrates* space.

If then a figure, such as a parallelogram or triangle, be reduced to its base (its altitude being reduced to *zero*), the surface of that figure will be reduced to *zero;* or the base having no surface, will be in that respect *zero; i. e., zero* of *surface*, or of area, which is measured surface.

A straight line whether it thus exist as the edge of a surface or be otherwise defined (e. g., the axis of a sphere), is yet *somewhat* in its own sense—in the sense peculiar to all lines—viz., in *length;* whereby, though it does not *divide*, it *penetrates* space.

A point, at the extremity of such a line, is not *somewhat* in *any sense*, but *only somewhere;* viz., at the very end of the line.

The like is true of a point, though otherwise situated; *e. g.*, at the centre of a sphere; where it is precisely at an equal distance from any and every point in the surface. This would cease to be true at any other position; though it were even at the smallest distance from the centre: so that this last cannot extend *somewhat* in any direction, nor yet be situated *any where else*, than in the position which has been already defined.

A point is thus the *absolute zero of space;* having "position, but not magnitude."

(9.) As in space there is room for the separate existence of all the material substances with which we are conversant, so, in duration, there is room, in a metaphorical sense, for the (successive) occurrence of events; and time is separated into portions, or has their termination marked by the limits of duration, as space is divided or limited by its bounding or limiting surfaces; or as a line is divided into distinct portions or is terminated by a point. When the one analogy will be the more complete, and when the other, will the more distinctly appear, when the infinities of both space and duration are considered in their proper connection. It will however be observed, that a limit such as the midnight with which one day (according to the ordinary reckoning) ends, and another begins, is not *somewhat* in *duration*, but only *somewhere;* or rather—if such a word were admissible—*somewhen;* viz., *when* the one day ends and the other begins. A limit such as this is *an instant;* and its relations to duration, or to that measured or at least finite portion of it, which we call time, are analogous to the relations of point to space. An instant is the *absolute zero* of *duration*, as a point is the *absolute zero* of *space.* [An instant is different, therefore, from a *moment;* which is a small but indefinite *portion* of duration.]

(10.) Rest, is, in a manner sufficiently analogous, the *zero* of *motion;* and may exist as the effect of an equilibrium; which is rest compelled.

This *zero* occurs, when *and* where, the body comes to, or is found at, rest; or when *and* where, it is prevented from moving.

(11.) An *equilibrium* is itself one form of the *zero* of *force;* though *such* a *zero* may simply imply the absence of *all* force, from a given *place*, and at a given *time: when* and *where*, there is *no* force.

(12.) *Perfect shadow* is the *zero* of *light; whenever* and *wherever*, it may exist.

(13.) Lastly. Empty space is itself the *zero* of *matter;* however great that space may be in capacity.

(14.) In any and all of such cases as have been specified, *zero* implies the absence of *that to which it is related;* and point being no extent in space; an instant, no time; rest, no motion; &c. Yet, an instant seems to be almost as incomparable with a point, as an hour with a mile. Being related to quantities entirely unlike in kind, each alone has place (in its own peculiar sense of the term) in that species of quantity of which it is itself the *zero.*

A point, however, may have place in a line, which itself, as before shown, may be a *zero* of surface, and this surface, again, a *zero* of capacity; for the point, the line, the surface, and that which the surface limits, are all to be found in space itself. But the other *zeros* which have been specified, an instant, rest, equilibrium, shadow, and (with reference to matter) empty space, though any or all of them may exist *when*, if not *where*, and sometimes even, *when and where*, all the *zeros* already described are found, yet the presence or the absence of one will not, in every case, imply, or require the presence or the absence of another; and the relations of all, as well as those of other *zeros* will, when carefully considered, lead to the conclusion, that *zero* is, in *no* case, to be regarded as the absence of *all quantity*—for then there could scarce be any occasion to consider it at all—but only as, in any case, *the absence of the quantity in question.*

[If then we should conclude, from considerations abundantly adequate, that before all that we here call somethings, there must have existed that which is not something; what we thus arrive at cannot be represented as a *zero*, except in the very respect that is *not something*, as the others are; but, even in so far as these considerations, thus exclusive, can determine, that which was before all these somethings may have been, and still may be *infinite* in its own way.]

Has Motion any place in Pure Mathematics?

(15.) What has already been said of a point, or the *absolute zero* of *space*, and rest, or the *absolute zero* of *motion*, may be found to have prepared the way for the consideration of the question:—how far, if at all, motion may be predicated of a *mathematical* point; or indeed, how far motion may have place, when what is to be moved, is a point, a line, a surface, &c., or any other quantity of those specially recognized by geometers.

Motion is *progressive* change of place. A body changes its place, as soon as it begins to move; *i. e.*, it *forsakes* the place which it occupied in space. It is transferred during the motion from place to place; and when the motion has ceased, the body is at rest; *i. e.*, no farther change of place occurs, but the body continues to occupy the place to which, at the end of the *progressive change*, it was transferred. The body itself was thus transferred, and not the place occupied by it: and all the boundaries, limits, or points situated in or about that space would be found to retain their positions, upon a reference to fixed standards. Neither space, then, nor the limits of it, are found to be the subjects of motion; that being, in so far as we can investigate it, a *physical* property of body, or, at most, of that which is in any sense connected with a body; as in the example of our own selves.

Yet a point, under certain circumstances, is, *as it were*, transferred along a line.

Thus when a pyramid so moves as to change the position of its vertex, the *mathematical* point at that vertex, is successively to be found at different places in the line which marks the limit of the whole space, either occupied, or passed through by the solid.

It should, however, be borne in mind, that a *mathematical* point, as already described, (8.) is not somewhat *in any sense*, but *only somewhere;* and the *place* of the point, in this instance, is precisely where the *pyramid ceases to be found at all*, and exterior space begins. Now as the pyramid, during its motion, continually *forsakes* the place it may happen at any instant to occupy, the point at the vertex being just *without* the pyramid, or at the limit of the space thus occupied, will be at once *left behind;* the motion by hypothesis being such that the vertex should not be stationary; i. e., the particular cases of a rotation about the vertex, without a progression from its position in space, being excluded. That the *mathematical* point at the vertex will be thus *left behind* or forsaken, will moreover appear, incontrovertibly, from the fact that its position as determined by fixed standards of reference will be found to be invariable.

It is nevertheless true, that the line in which, or precisely *at* which, the vertex, during the motion, is always found, will be distinctly marked out; it being the limit *up* to which that space extends, which was either occupied or passed through by the pyramid.

The like must be true of the centre of gravity of a sphere in motion through space, or which has so moved. A *new* point in space will be found to be the position of the centre of gravity, as the sphere advances. Still more obviously must the like be true of the centre of gravity of two or more bodies, when they so move as to change its position, that centre moreover being throughout supposed to be without the bodies themselves. When the masses, &c., of the bodies are known, the successive positions of the centre of gravity of two or more may be computed, or even prescribed; yet such a mere position, at any instant in space, is not *pushed* forward or *drawn* backward by or with those bodies; and all this, while, moreover, the entire curve in which all the successive (but certainly *different*) positions of the centre of gravity are situated, may throughout admit of being accurately defined, and its limits therefore precisely settled. Indeed, lastly, should we suppose the contrary to all this to be true, we could not escape from the seeming contradiction, that a point which (8.) is the *absolute zero* of space should become *somewhat* in space, that is, should be drawn out into a line which has length, by the introduction of the *foreign* element of motion.

We must, in view of all that has been advanced, regard the motion of a *mathematical* point, as a pleasant fiction; the result as regards position, magnitude, and of the quantities concerned, being the same as it would be if such motion were possible; while the actual description of a *mathematical* line in space would require the motion of *a pointed atom*, if such a thing may be.*

* This does not militate against the *mathematical* existence (1.) of such curves as the cycloid, &c.; since it is only necessary to suppose the generating circle, or other curve, &c., to be drawn on *a material substance;* that it take successively the several positions required; and that the point at the edge, or elsewhere, be assumed successively where the describing point ought to be.

This fiction may not however be wholly harmless, when not merely *mathematical*, but also physical relations are the subjects of investigation. Such is the case in mechanics, when motion becomes the subject of investigation as a *physical* property, as which alone, in accordance with what has already been said, is it in any case (in effect) to be regarded. The motion of a body involves then the motion of its atoms, and as there can be no moving *mathematical* point, a moving point, or whatever in mechanics may be spoken of as such, must be a *moving atom*. Its physical property of motion, the laws which regulate it, and those which determine an equilibrium, must be made to rest upon observation, experiment and induction.*

Corollary.—There can be no "*Rational Mechanics*," in the sense in which that phrase is often employed.

The considerations already urged, against the doctrine of the motion of a *mathematical* point, will apply with equal force to the case of a *mathematical* line, surface, or solid.

Fundamental Reason for the Existence of Incommensurability.

(16.) From the consideration of limits, zeros, and their special relations, we may pass to that which supposes the introduction of new limits; viz., the division of quantities into parts or portions; fractions, properly so called, among the rest; by the aid of which, the nature of incommensurable quantities and the necessity for their existence may both be made apparent.

If we select as a very simple example, a finite straight line; and suppose it to be divided in the middle, into its two most simple fractions; viz., its two halves; each half will, of course, be equal to the other. When we divide the same line into thirds, three fractions will be obtained, all equal among themselves. The same perfect equality of the parts will still be found when we successively divide the line into fourths, fifths, &c.; any one of such fractions being an *aliquot* part of the whole; and any fraction, such as $\frac{3}{4}$, $\frac{5}{6}$, &c. of the line, a combination, or grouping together, of such *aliquot* parts.

Now, however many such divisions of the whole into all the several fractions of the series of halves, thirds, &c., may be made, it must happen, if the process be far enough continued, that some of the points of division will agree, ($\frac{3}{4}$ of the whole being equivalent to $\frac{9}{12}$ of the same, &c., &c.;) and there no new division of the line will take place. Yet some must also differ, at each new division; since $\frac{1}{2}$ cannot $= \frac{1}{3}$, nor $\frac{1}{3} = \frac{1}{4}$, &c.; and very many other combinations, such as $\frac{2}{3}$, $\frac{3}{4}$, &c., must be different, as the

* The French phrase, "un point matériel," is descriptive of the real state of the case; whatever may be said of the reasoning in connection with which that phrase may sometimes occur.

theory of numbers would indicate. *However many divisions* into *aliquot parts* may then be effected, there always must be positions on the line, included between the points of division so obtained; since no two of the successive divisions of the whole line can agree, as has already been shown, at *all* points. In other words, there must exist positions between the points of division so obtained, which no division of the line into fractions, (i. e., *aliquot* parts or their aggregates,) *however numerous*, can ever mark. At any or all such positions, the line would be divided into two parts *incommensurable with the whole*, and of course *incommensurable with each other.*

The combination, by addition, of the original line or unit and a line equivalent to such a portion, would be of a length which may be represented as between 1 and 2 such units, but the excess above 1, such as cannot be expressed by any fraction, &c., &c.

The like principles must be applicable to the case of any other quantity which will admit of the like successive fractional division.

The fundamental reason for the existence of incommensurable quantities seems, then, to be, more concisely, this: The division into fractions is a division into *aliquot* portions, or implies such a division of the whole as the aggregate of such portions would furnish. Now this is so far from being the only mode of originally dividing the quantity, that it must be regarded as a peculiar and restricted one; in so much that it would almost seem that the result of a fractional division is not that which would most probably be obtained, if the quantity were divided at hazard; or the chances would be more numerous, that the quantity would be divided incommensurably, than that it would be divided into fractions properly so called.

(*To be continued.*)

Art. XVI.—*Notes on the Geology of Charleston, S. C.;* by F. S. Holmes, Corresponding Member of the Acad. of Nat. Sciences, Philadelphia.

That Charleston, the Capital of South Carolina, is built upon geological formations identical in age, and in other respects similar to those upon which the great cities of London and Paris are located, is a curious fact but lately ascertained. The basin shaped depression of its underlying calcareous and other beds, as determined in the survey just made by Professor Tuomey, occupies a considerable extent between the Savannah and Peedee Rivers, and rests upon an older group of rocks known to geologists as the Cretaceous formation. The sides of this basin are estimated to be of sufficient inclination to produce those artificial fountains,

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Art. XXVII.—*Notice of, and citations from a Voyage of Discovery and Research in the Southern and Antarctic Regions, during the years* 1839–43, by Captain Sir James Clark Ross, R. N., Knt., D.C.L. Oxon., F.R.S., etc.; with plates, maps and wood-cuts. In two volumes, 8vo, pp. 366 and 447. Lond. 1847.

Voyages of discovery are among the most interesting and important of the adventures undertaken by man. They have been prosecuted in all ages since the introduction of the mariner's compass, and have been particularly numerous since the middle of the last century. England, France, Russia, and recently the United States, have sent forth exploring squadrons, as well as expeditions by land; their ships have traversed all the great oceans, and have pushed their daring voyages far within the arctic and antarctic circles, amid seas covered with floating icebergs, and in close proximity to the eternal barriers that repel any nearer approach to the frozen poles. The expedition under Capt. Wilkes, which certainly ranks among the ablest and most interesting of these undertakings, we have had frequent occasion to mention with warm approbation. It has done honor to our country, and will ever remain a memorable and illustrious event in its history.

Passing by other recent voyages, we propose for the present, to confine ourselves to the Antarctic Expedition of Sir James Ross; and as this work has not yet been reprinted, we may notice it somewhat in detail. This voyage arose from the recommendation of the eighth meeting of the British Association, held at Newcastle in August, 1838. The principal object proposed was the extension of physical science, especially in relation to terrestrial magnetism, to the importance of which the attention of the Association was invited by Lt. Col. Sabine, and it was enforced by

Mount Terror being much more free from snow, shewed numerous little conical crater-like hillocks, each probably (like those on the flanks of Etna) once in action. Lofty ice cliffs, probably over 1000 feet in thickness, solid without a fissure, and presenting vertical walls to the waves which dashed their foam high against them, stretched away interminably to E.S.E., and the ships sailed along them more than 100 miles without any prospect of passing around them. The latitude was now 77° 46′ S., longitude 176° 43′ E.

The magnetic dip had diminished to 87° 22′ S., proving that they had passed beyond the magnetic pole, and the variation was 104° 25′ E. Soundings were obtained in 410 fathoms with two feet of soft green mud; temperature at 300 fathoms, 34° 2′, in the air 28°; in summer the air and the water seldom differ more than three or four degrees.

A petrel wounded by a shot, falling in the water, was immediately torn to pieces by its companions.

At midnight the lat. was 78° S., in 180° E. long.

Magnetic irregularities.—In lat. 77° 6′ S., long. 189° 6′ E., the dip had diminished to 86° 23′, the variation decreased from 96° E., to 77° E., and then again increased to 16°. It appeared to be one of those extraordinary magnetic points first observed during Sir Ed. Parry's second voyage to the Arctic seas near the eastern entrance of the Hecla and Fury straits.

The highest latitude attained this season was 78° 4′ S., and all progress farther south was prevented by a barrier of ice 160 feet high, and extending in one unbroken line 250 miles.

(To be continued.)

Art. XXVIII.—*On the Fundamental Principles of Mathematics;* by Stephen Alexander, Professor of Mathematics and Astronomy in the College of New Jersey.

(Continued from p. 187.)

Of Positive and Negative Signs, and a Particular Case of Imaginary Values.

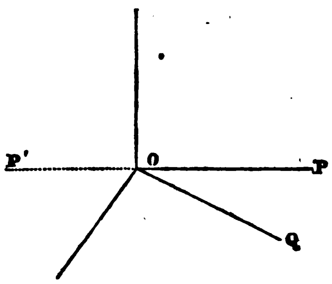
(17.) The consideration of the algebraical signs of quantities seems to be next in order, after what has already been exhibited, with regard to their division; for the relative greatness of the portions obtained, and the *manner* in which the greatness of the quantity will be affected by that of another in any given case of combination, will depend upon their respective signs; while neither the absolute, nor yet even the relative greatness of the quantities, will be affected by those signs.

Thus -2 indicates the subtraction, or at least the negative relationship, of *neither more nor less* than $+2$ (the negation of neither 3 nor 1 would answer the purpose) and $-Q$ indicates the subtraction, or at least the negative relationship of *as much as* $+Q$, *and no more.* In neither case can one of these quantities produce in a like combination a greater or a less effect than its own negative would produce; but that negative in such a combination (*because* it is *negative*) will be *destructive* to *precisely the same* extent to which the positive quantity is *constructive.* Hence, a negative quantity ($-Q$) cannot be regarded as being *in fact* less than *nothing* (i. e., (14) as less than the *zero of its own species*), but only as being *in effect* less than the zero of the species, in one particular respect; viz. *that very respect, in which the other quantity* ($+Q$) is *positive.* For *the zero* is *nothing* in that *specific* respect; while the negative quantity ($-Q$) is precisely as great indeed as $+Q$, but of an opposite *character* in the *very respect* in which $+Q$ is positive; insomuch that $+Q$ would be precisely destroyed by $-Q$; i. e., annihilated or reduced to *the zero of the species.* Or (as viewed in the opposite direction) $+Q$ would be precisely adequate to the destruction of $-Q$, reducing *it* to *the zero;* and a second similar introduction of $+Q$ would, in place of this last result of *the zero*, give $+Q$ itself: or, it appears, that $-Q$, in *the very respect* in which $+Q$ is positive, is *in effect,* as much below *the zero of the species*, as $+Q$, in the positive sense, is above it.

Similar principles will be applicable to the results of like combinations of other quantities with the respective quantities, $+Q$ $-Q$ and *the zero;* the respective results being represented by $+fQ$, no result, or *zero* (*of the same species with* $+fQ$,) and $-fQ$.

In the determination of the position of a point in space, reference, as is well known, is made to three coördinate axes, all meeting at one point—the origin. If from this origin O, we measure outward upon any of the three axes, we naturally mark the measured length as positive; since it increases as we proceed in that direction in space. If we measure from P toward O or P′, any distance less than PO, the quantity thus measured will thereby be taken from PO or will have an effect, the negative of the previous increase. If we thus measure from P a distance equal to PO; this distance will extend to the origin; and PO will be subtracted from itself, leaving no remainder; i. e., PO will be reduced to a point, or the *zero of*

Fig. 1.



length. If (still in the same direction) we measure from P a distance such as PP′, greater than PO, the farther extremity of the line so measured will extend *beyond* O to P′, i. e., to a distance equal to the excess of PP′ above PO; or the attempt to subtract from PO a quantity *greater than itself*, will result in a *negative* remainder equivalent to the excess of the greater quantity PP above PO; and this remainder will extend from O itself, in the direction *opposite* to that of OP. If PO = *zero*, the *whole* of PP′ will in this manner, extend from O in the negative direction.

The negative quantities thus originating are, in all the respects specified, strictly analogous to those which present themselves when the attempt is made to subtract 7 from 4, or 9 from 6, or, in general, the numerical quantity n from q, when n is greater than q;* the differences in the results of such subtractions being no other than those which must exist in the case of quantities of another species.

The same reasoning will apply to distances measured, in like manner, upon either of the other axes. Hence distance outward from the origin, in the direction first assumed, will in any case be naturally positive; and distance in the opposite direction, negative; and will be exhibited in its *isolated* as well as negative character, when measured in that direction, beginning at the origin.

These conclusions being independent of any particular inclination of the axes among themselves, will apply to the case of three axes the sum of whose three angles—that of first axis with second, second with third, and third with first—differs scarce at all from four right angles; and this, whether those three axes be situated on the one side or the other of a given plane of reference, passing through the origin O. As, therefore, the conclusions referred to will be applicable, however near the state of things may approach to that in which the three axes would be all in one plane, and this, on either side of that state of things as a limit; these same conclusions must be regarded as true in the case of that limit itself: or direction from the origin outward must be regarded as positive, whichever of the three axes may be employed to indicate it, and the contrary direction be regarded as negative, *even when carried beyond the origin by excess of distance extended in that direction:* the three axes being moreover all in the same plane.

As, moreover, the conclusions, from first to last, are independent of any fixed direction of one or more of the axes, they will all be alike applicable to any other three axes, which like OQ do not coincide with any of the first three; direction from

* The view here presented will be found to coincide with that of *M. Faure*—"*Essai sur la Théorie et l'Interprétation des Quantitiés dites Imaginaires, Premier Mémoire*, (16.) *a Paris*, 1845." As M. Faure moreover intimates [*Essai, &c.*, (19.)], the distance subtracted from OP may be regarded as measured *negatively* from a *new origin* at P; precisely as OP′ is measured *negatively* from O.

the origin outward being still positive and the contrary negative, &c., in the case of any such axis; all the six axes being moreover in the same plane. What has thus been extended from the case of three axes thus situated to more than three, may in like manner be extended to any number of axes however great. Hence direction outward must be regarded as positive on any and every straight line drawn from the same origin, in the same plane, and the contrary direction must be regarded as negative, even when it extends, *after that manner, on the contrary side of the origin.*

We have, thus, exhibited *the foundation for the analytical necessity of regarding a radius, or a radius-vector, as positive, when measured outward from the centre, or pole; but negative when measured in a contrary direction, even when, after that manner, it is regarded as extending across or beyond that centre, or pole.*

(18.) The doctrine of "imaginary quantities" would be next in order; but this, of itself, would furnish matter for an entire dissertation; if it were even advisable to enter upon the consideration of a subject, so much and so often discussed.* It may not however be amiss to advert to one or two results of analysis which seem to admit of explanation, by a reference to the principle, that imaginary quantities, occurring in a geometrical investigation, may sometimes have a possible existence out of the plane of reference. Two equations first discovered by Euler,

$$\sin. x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{1}}, \text{ and } \cos. x = \frac{e^{x\sqrt{1}} + e^{-x\sqrt{-1}}}{2},$$

when transformed, by substituting for the real arc x, the imaginary arc $x\sqrt{-1}$, give, respectively,

$$\sin. (x\sqrt{-1}) = \frac{e^{-x} - e^{x}}{2\sqrt{-1}}, \text{ and } \cos. (x\sqrt{-1}) = \frac{e^{-x} + e^{x}}{2}.$$

Here the cosine is real, though the sine and the arc are both imaginary. This seems to arise from the fact that the cosine might, *in effect*, be found in *the common intersection* of the plane of the imaginary arc $x\sqrt{-1}$, and that of the axes of reference. It, therefore, has a real value; while the sine and arc, being both out of the plane of the axes, are imaginary. This being admitted, the secant of the real arc (i. e., the arc whose cosine has this real value) will $=1\div$ by the value of the cosine; while the secant of the imaginary arc having the same cosine must, it would seem, $=\sqrt{-1}\div$ by the value of that cosine; unless the imaginary arc were reduced to the limit of 0°, or 180°, or 360°, &c.; when it would be terminated in the plane of the axes: when also its sine must $=0$.

* See among others, the "Essai" of M. Faure, (already cited in (17.).)

Of Infinites of Various Descriptions.

(19.) (*a.*) We shall designate a quantity as being ***absolutely infinite;*** if it be so great as to be ***utterly boundless or destitute of any limit.***

This is the case with "absolute space;" which whether we regard it in a direction forward, or backward, or upward, or downward, or sidewise, or obliquely, is, in any and every direction, positively boundless or absolutely infinite.

So too, far back as the imagination can extend—antecedent to all ages past, antecedent to the existence of all created beings or things—we still behold, self-sustained on the throne of his adorable perfections, the GREAT FIRST CAUSE; who being the very origin of the first beginning, HIMSELF has none; but ever was, as now, "*from everlasting.*" It is in this underived antecedent, this perpetual precedent, of the *Divine Pre-existence*, that we find the realization of *Eternity Past.*

For also beyond the ages to come—unmeasurable though they may be "by the flight of years"—must still endure the ceaseless and unalterable being of HIM "who alone hath immortality" underived: and, *in that*, is Eternity Future. Yet what mental vision shall penetrate the "clouds and darkness" which surround the Divine Pre-existence; and inform us how it was, that, in Eternity Past, time e'er began. Or—fixing its unfaltering gaze upon "the light inaccesible" which covers, as with a veil of "glory," the designs and capabilities of the future—say what means the duration of an immortality once begun in his presence? Yet is it the combination of both these, of that eternity which always was, and that which "ever shall be"—nothing less than this, nothing short of it—that constitutes *the absolute infinite of duration?* It is the inexhaustible fullness of the being of HIM "who inhabiteth eternity," in its twofold sense; having ever been, as now, "*from* everlasting *to* everlasting."

(*b.*) We shall designate a quantity as being *specifically infinite,* if it be as boundless as those already described, ***in certain respects only.***

Thus if a straight line be without termination, in either direction, from a point which might be assumed in that line, such a line will be specifically infinite; viz., in length—in which respect alone a line can be great or small. In this same respect might a surface be said to be infinite, on which such a line could exist, or the solid within or on which such a line would be possible: whatever might be the other dimensions of either the surface or the solid.

If a line, alike interminable, were any where curved, such a line must be regarded as *longer* than the other; since it would intrude upon what may (by indulgence) be termed the ***breadth*** of absolutely infinite space, as well as extend through what, for want of a better term, must be called its length.]

A surface without border would be specifically infinite in length, breadth, and a contiguous and plane superficial area.

[A surface without border and altogether plane, must, notwithstanding, be regarded as less than another surface, which though alike without border, yet deviates any where from a plane; since the latter not merely extends through all space in every direction which can be called length and breadth, such as exist upon a plane, but also encroaches upon what we may, in such a comparison, term the *thickness of space.*]

The quantities here characterized as specifically infinite are innumerable; and some may be parallel to one another: while, in so far as we can discern, there is but one absolute space; i. e., *one absolute infinite of extension.* So also there is but *one absolute infinite of duration.*

(*c.*) We shall designate a quantity as being, in comparison with another, *relatively infinite;* if the ratio of the quantity to that other be too great to be expressed by any assignable number, however large.

Any assignable number, however large, may be exceeded by the continued addition of the number 1 to itself; and then again to the sum, &c. &c.; and the like must be true with regard to any series or aggregate of the units of any species; when the number of units is assignable. But if, instead of continually adding the original unit, or its equivalent, we take its double, and of that product *its* double, &c. &c., and continue the process of successive doubling, until the *number* of such individual *processes* is as large as any number which we can assign; the aggregate will far exceed that obtained by successive additions, repeated as often.

For in the one series, the quantity to be added, at each successive step, is constant; so that if Q denote the original unit, the aggregate of the series or

$$S = Q + Q + Q + \&c. \ldots\ldots ;$$

but in the process of continued doubling, each term consists of the aggregate of all that preceded, added to as much as itself; and therefore the sum of such a series, or rather the resulting aggregate,

$$S' = Q + Q + (2Q) + (4Q) + \&c. \ldots\ldots ;$$

the terms after the second continually increasing. If then the number of *terms* in each series be as great as any that we can assign, the number of *times* the original quantity Q contained in the aggregate of the second series will be *too great to be assigned;* and will in any case exceed the number of units such as Q, which we may assign to the first series, however great that number may be. An aggregate such as S′ would then be relatively infinite in comparison with Q.

The like must, *a fortiori*, be true, if, in any or all of the processes of successive multiplication, the multiplier were more than 2; so that the multiplicand would be more than doubled.

If, by an inverse process, there were taken from Q its half, and then from the remainder its half, &c. &c., a sufficient number of times, we should, in the end, obtain a quantity so small, in comparison with Q, that no multiplier of it could be found sufficiently large to reproduce as much as Q.

For in this case, if the *number* of individual *inverse processes* were equal to that of the direct processes in the former case, and L be the last remainder; then, beginning with L, we must, in effect, repeat the process of continued doubling as often as before, in order to reproduce as much as Q; or Q will itself be relatively infinite in comparison with L; or L will be "an infinitesimal" in comparison with Q.

The like must be true, *a fortiori*, if at any step in the process, more than half were removed.

As, moreover, Q is relatively infinite in comparison with L, and S′ again relatively infinite in comparison with Q; so again, by continued doubling, beginning with S′, might another aggregate be obtained, which would be relatively infinite in comparison with S′, &c. &c. On this it is unnecessary to dwell; as one mode of exhibiting the differential calculus, owes its peculiarity to the employment of quantities such as these.*

It is important however to observe, that this description of infinity is the only one which can be predicated of *number*, *velocity*, *mere mechanical force*, &c. &c.

For no number can be so great, that a sum of units *might not exist*, (1.), which should exceed that number.

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* It may not be amiss, here, to notice an argument against the consistency of the results of mathematics which may be thus exemplified. An inch may be divided, and the remainder subdivided, &c., by the process already explained, and thus the infinitesimal of an inch obtained; and the aggregate of all such infinitesimals into which the whole inch might be divided, would be equal to the inch itself. Now if the inch were passed over by a moving body, the passage over each infinitesimal would occupy some portion of time. But the number of such portions of time would be infinite; since the number of the infinitesimals of the inch is infinite. Hence (says the objector) it must require an *eternity* for a body to move over an inch—which is absurd. The conclusion is indeed absurd, but *that* conclusion follows not from the premises. For as the inch is relatively infinite, i. e., infinite *in comparison* with its infinitesimal, in the restricted sense of relative infinity; but still finite and capable of measurement by a comparison with other standards; so the time requisite to pass over an inch with a uniform velocity, will be relatively infinite, i. e., in comparison with the time in which the infinitesimal of an inch would thus be passed over; while *it* also might be finite and capable of accurate measurement by a comparison with another standard. This portion of time then could only be called a *relative* eternity; if that were not an abuse of the term. If the whole inch were traversed in a minute, this relative eternity would endure but for a minute; while it would still be true that the infinitesimal of an inch would be traversed in an infinitesimal of that minute.

And no velocity could be so great, that the body moved would be in two places, at the same instant; for that would contradict what all experience has shown to be true of the nature of body. Hence the transfer of a body from one place to another, however rapid, must occupy some time: and it is mathematically *supposable*, and, for aught that can be discerned, physically possible, that the body might be made to pass through or over a greater distance in the same time; i. e., any velocity, even that which is too great to be measured, might admit, it would seem, of an increase.

So also, however great a mechanical force may be applied in any case; another might (for aught that can be discerned) be added to and combined with that force.

(20.) It may be observed in brief that the three descriptions of infinity obtain respectively, thus:

1st. *Absolute infinity*, when the quantity is so great that there is no limit to it.

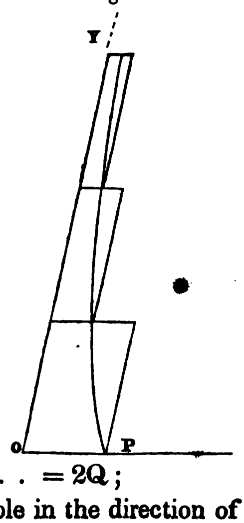
2d. *Specific infinity*, when this boundlessness exists in certain respects only.

3d. *Relative infinity*, when the one of two quantities of the same species is too great to be measured by the other.

Of Finite Quantities which are specifically infinite in one Dimension.

(21.) The results of the *Integral Calculus* have long since indicated that certain areas whose *limits* in part are lines *interminable in one direction*, may yet themselves be finite. Such are the areas bounded, in part, by certain curves and their asymptotes. An area may also exist having for its partial limits straight lines, and on one side a line interminable in one direction, or even in both directions, and thus, (19.), specifically infinite; and yet, as it would seem, be finite in surface. Such an area will exist, if the arrangement of its portions be that represented in the figure; each parallelogram having its sides in one direction equal, each to each, to those of any other which are situated in the same direction; but each having its sides, respectively, in the other direction, but one half of the length of those which immediately precede them, in the series. Then if Q be the area of the first parallelogram, or that represented as lowest in the figure, the sum of the "infinite series" or

Fig. 2.



$$S = Q + \tfrac{1}{2}Q + \tfrac{1}{4}Q + \&c. \ldots\ldots = 2Q;$$

the line OY being supposed to be interminable in the direction of

Y. If a similar construction should exist downward, or in the opposite direction, the sum of the areas of both would still be *finite;* it being equivalent to 4Q; but the limit of the surface along OY would be a straight line, (19.), *specifically infinite.* This last must be true, since S would still differ from 2Q, (being an infinitesimal less than it,) if OY produced were only, (19.), relatively infinite: it will be equivalent to 2Q only in case the border or limit OY really have no termination in the direction of Y.*

Comparison and Contrast of a Finite Quantity with the Infinite of its own Species. Relative Zero.

(22.) The distinctions of the various infinites having now been exhibited, we may be the better prepared for the comparison of a finite quantity with an infinite of its own species.

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* If a curve be drawn as in the figure, this curve will be the ordinary hyperbola, and OY its asymptote. Now the *Integral Calculus* will indicate that the area bordered by OP, OY, and the curve is *not finite* when the two latter are interminable in the directions in which the approach of the one to other takes place. Yet this area is less than that of the other surface already described; a portion of that other surface being left out by the construction of the curve: i. e., the area bordered by the curve is less than 2Q; or *it must be finite.* Here, then, is *a paradox.* May it not be true that in this case a concealed term exists in *the constant* which must be introduced, in the integration; especially since the equation applicable to this case, if integrated according to the rule for the integration of differential quantities containing a power of the variable; will exhibit infinity in the result: it being in fact the excepted case; which however may be made to exhibit a finite result, when integrated by the aid of logarithms.

The equation of the hyperbola, the asymptotes being the axes, is

$$xy=\frac{A^2+B^2}{4}=q. \text{ Hence,}$$

$$x=\frac{q}{y}, \text{ and } dx=\frac{-qdy}{y^2},$$

$$\therefore ydx=\frac{-qdy}{y}=-qy^{-1}dy;$$

which is the excepted case.

M. L'Abbé Moigno (*Leçons de Calcul Différential et de Calcul Intégral,—Calcul Intégral,* 1re *Partie,* 16,) disposes of the excepted case, in the general, thus: Le second membre de la formule,

$$\int x^m dx=\frac{x^{m+1}}{m+1}+C$$

semble devenir infini pour $m=-1$; mais comme on peut l'écrire sous la forme

$$\int x^m dx=\frac{x^{m+1}-a^{m+1}}{m+1}+C;$$

il devient réellement indéterminé; on obtient sa veritable valeur en prenant le rapport $x^{m+1}\log._e x - a^{m+1}\log. a$, des dérivées du numérateur et du dénomenateur, et y faisant $m=-1$, ce qui donne

$$\int x^{m-1}dx=\int\frac{dx}{x}=\log. x-\log. a+C=\log. x+C,$$

comme on le sait *à priori.*

If a point (P) be assumed in a straight line, (19.), specifically

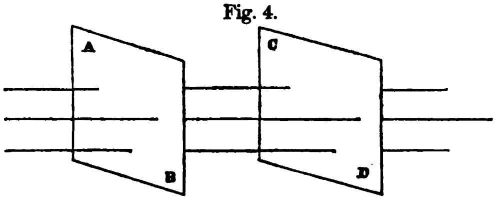
Fig. 3.

infinite, the line in either direction from that point will be interminable; and the two portions (if they may so be called), one on each side of the point, may be regarded as being, *in effect*, equal. If, again, another point (P′) be assumed in the same line, however remote from the former, the two portions, one on each side of it, may again be regarded as being, in effect, equal; though the whole intervening distance (PP′) will have been added to one of the portions into which the line was divided at the first point (P) and subtracted from the other portion. It appears therefore that any such distance, *however great*, must be regarded as *nothing* in comparison with a straight line interminable in only one direction.

Similar reasoning applied to the case of any other of the specifically infinite quantities described in (19.); (or rather to what in like manner may be regarded as being in effect their halves,) would lead to a similar conclusion with respect to them.

If moreover a plane without border be extended any where in space, all that region of absolute space on the one side of the plane must be regarded as being, *in effect*, the half of all space, and all that region on the other side of the plane, as being, *in effect*, the other half of the same. But the like will be true of the regions found, the one on one side, and the other, on the other side of a plane of the same description, parallel, it may be, to the former, but at any distance from it, however remote; the one

Fig. 4.



being without border and coinciding, in direction in space, with AB; the other alike without border, but coinciding, in direction in space, with CD. But if the dividing or separating limit be at one of these planes, instead of the other, all the intervening space will, as it were, have been added to the one half of all space, and taken from the other. Yet the two regions which are separated by the second plane of which CD is a part, are still to be regarded as being, *in effect*, equal, or each as still, *in effect*, the half of all space.* Hence, all the intervening space separated from the

* In this as in other instances it will be observed, that the truth arrived at, in so far as the so-called halves are concerned, admits of being otherwise illustrated. When a straight line such as PP′ in the figure at the commencement of this article (22.) is finite, the middle is at an equal distance from each end.

But when the line is interminable in both directions, there is no extremity in either direction to measure from, and thus determine the middle. The middle rest by two planes without border, must be regarded as *nothing* in comparison with the so-called half of all space; though in the space so separated there would be room for all the visible creation, could its form be adapted to the dimensions of that space in those respects in which the dimensions are finite: or the two planes might themselves be supposed to be situated outside of all that we can discover by the best optical aid; and the conclusion still be the same; viz., that the whole space separated by them must be regarded as nothing in comparison with all space; which is itself absolutely infinite.

In the space thus separated, might also exist all the specifically infinite quantities described in (19.)

As the point assumed in the interminable line, *in effect*, divides that line into two halves; and the plane without border in like manner divides all space, so (19.) the instantaneous present divides the Eternity Past from the Eternity Future. It does so (in so far as can be discerned) in all worlds at once; as the same plane in the figure cuts all the three straight lines which penetrate it, and which are to be regarded as interminable in both directions from the dividing plane. The present thus divides those two Eternities now. So, also, it was, in so far as can be discerned, after the first moment of the existence of the first created being or thing, and thus it shall be, after the present system of things, like a worn out "garment," is, as it were, "folded up" and laid aside.

Through the limit thus ever present the current of time passes, in a metaphorical sense; and, moment by moment, the Eternity Future is transferred to the Eternity Past. [We seem to recognize this even in our ordinary language. Thus we say, when tomorrow comes (viz., to us) and not when we come into to-morrow.] As, moreover, in the case of the analogous quantities in space, so, in this case, the transfer, whether of a single day, or of countless ages, from the one Eternity to the other, will be found to leave each of those Eternities *in effect* the half of, (19.), the absolute infinite of duration. Hence in manner as before, all the

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being thus actually indeterminate, may exist, in so far as it exists at all, any where in this line.

The like may be said of all space, which has no borders. Or—with reference to its boundlessness on all sides—that its centre is any where.

A line interminable in one direction may, as already intimated, be regarded as the half of the line interminable in both directions;—but it does not seem to be possible to obtain in the same form any quantity which could be called the one-third, or the one fourth, &c., of the whole. Yet if an interminable curve be supposed to exist of such a form that it might throughout meet the interminable straight line, and, any and every where along it, be found finite portions of the curve, each equivalent to $1\frac{1}{2}$ or $1\frac{1}{3}$, &c. times the corresponding fraction of the straight line, a fraction such as $\frac{1}{2}$ or $\frac{1}{3}$ of the interminable straight line would seem to exist combined with the whole and all expressed together by the interminable curve. It would *seem* to be so; but any such conclusion should be received with caution.

intervening portion of duration thus transferred must be regarded as *nothing* in comparison with either Eternity.

We may thus in some very humble measure learn how it is, that, in the view of the INFINITE MIND, "a thousand years" should be "as one day, and one day as a thousand years."

(23.) A finite quantity in comparison with others of the same species, which are in some respect boundless, has, (22.), been found to be as nothing. We shall therefore designate it in this comparison as *a relative zero;* it being *zero by comparison*, in a more *intense* sense, than the quantities described in (19.) were *relatively infinite*, in *comparison* with others beneath them; and also a more *intense* sense than the same quantities were *infinitesimal*, in *comparison* with those above them.

Character of the Symbols $\frac{1}{0}$ *and* $\frac{0}{0}$.

(24.) If the straight line described in (22.), as interminable in one direction, be assumed as a measuring unit, then any finite straight line being, (23.), a *relative zero*, we shall have for the symbol of the ratio of the greater of these quantities to the less $\frac{1}{0}$; and for that of the less to the greater, $\frac{0}{1}$.*

Since, moreover, the finite straight line is a relative zero, (the line interminable in one direction being the standard); if p denote the length of one finite straight line, and q that of another, we shall have in manner as before,

p represented by or $=\frac{0}{1}$; and q represented by or $=\frac{0}{1}$; whence,

$$\frac{p}{q}=\frac{0}{0}.$$

Or more directly still; p being *relatively* represented by *zero*, and q by the same,

$$\frac{p}{q}=\frac{0}{0};$$

as before.

From this equation, however, neither p nor q can be determined; nor even the *actual ratio*, of the one to the other.

* The line interminable in one direction is, moreover, either the secant of the tangent of 90°; or, when taken negatively, it is either the secant or the tangent of 270°; and radius of the circle being 1, we have tan. $90°=\frac{\sin. 90°}{\cos. 90°}=\frac{1}{0}$; and sec. $90°=\frac{1}{\cos. 90°}=\frac{1}{0}$; results agreeing with the preceding determinations, when the greater line was regarded as *the unit*, and the less became *a relative zero*.

The indeterminate character of the symbol $\frac{0}{0}$ is thus established by primary considerations.

[In the case now described the symbol or form $\frac{0}{0}$ presented itself, because the terms were *zero* in *comparison* with a standard unlimited in at least one respect. In the instance of a fraction such as that represented in the equation,

$$\frac{Fx}{fx} = \frac{P(x-a)^m}{Q(x-a)^n}$$

which, when $x=a$, becomes,

$$\frac{Fx}{fx} = \frac{P\times 0}{Q\times 0} = \frac{0}{0};$$

the form or symbol $\frac{0}{0}$ appears, because the particular value of x reduces each of the multipliers, $(x-a)^m$ and $(x-a)^n$, to *zero.* Hence *no product* can, in effect, result, whatever may be the value of the multiplicand; i. e., (14.), *nothing* will be found in the place of the numerator as well as that of the denominator: or the value of the fraction, in its form of $\frac{0}{0}$, becomes indeterminate, not as in the former case, because of the character of the standard of reference, but because of *the actual disappearance* of every thing from both terms of the fraction, which under other circumstances, could render them *definite.*]

Another Application of Preceding Principles.

(25.) [The relations of things being, as already maintained in (4.), *constituted relations;*—and they also being constituted in some respects alike, as appears from the comparisons between those of space and time in (22.)—we may even reverentially proceed a step farther, and conclude, that, as any finite (or even in some respects boundless) space is *worthless,* or to be regarded as *good for nothing, in comparison* with the *absolute infinite of space;* and as, again, any finite portion of duration is also *a relative zero,* in comparison with *the absolute infinite of duration;* so, also, must the highest created intelligence and lowest among men be *alike worthless* or regarded *as nothing* in comparison with the ALONE INFINITE ONE: or man, placed as it would seem lowest in scale of such intelligences, must be represented, in comparison, as being as it were "less than nothing." This truth has important moral bearings; but this presentation might be regarded as out of place in a mathematical dissertation.]

Wherein Necessary Truth is to be found, and the Final Hypothesis.

(26.) The consideration of the two great relations of things, duration and space, has often prompted the question whether, if the visible universe were annihilated, space would remain?

On the one hand—since the limits of things actual are, (8.), no part of those things; but their surfaces, as connected with the things themselves, bound them and are removed whenever and wherever those things are transferred; yet cannot be removed *from* the things, and thus placed by themselves—it would seem, from all this, that what is less a relation of things, would, if those things were gone, not exist *by itself;* or *no longer be.*

On the other hand, surface, &c., are *dependent* relations of things—as must appear from what has just been stated—while space is independent of them, in so far, that when the thing is removed the space, (15.), which it occupied, is forsaken or left behind; and may be again occupied by something else. Hence, it would seem that, if all such things were gone, space would *still be.*

Certain it is, moreover, that, in so far as we can discern, if the visible creation were annihilated, there would be room for another. But does this conclusion amount to anything more than the assertion, That, under the new system of things, space would again exist as a relation of them; as it does now? If so, then this cannot determine what would be, if there were *no* such things. It appears then that such a state (or rather absence) of things is so far without the pale of our experience, that we can form no adequate idea of it; and must therefore leave the question of the existence or non-existence of space, in the absence of such a creation as we now have, without an answer.

But whatever the reply to that question ought to be—space exists not, nor can it exist independent of the GREAT FIRST CAUSE, who formed all things, and "by whom they also *consist;*" with whose existence, moreover, the absolute infinite of duration, (19.), is interwoven.*

Could space, indeed, exist independent of HIM, or does it so exist, then it exists not of "his good pleasure"—then was it not *created*—then must it be *self-existent*—but then must it, also, be found *in* HIM; which contradicts the hypothesis: and that hypothesis must therefore fail.

We know therefore of no space which is not pervaded by his presence, as we are certain that there is no duration which "He inhabiteth" not.

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* If it were imagined that *duration* might exist, *though* THE FIRST CAUSE *were not,* —the reply must be that the hypothesis of HIS non-existence is itself the greatest possible absurdity—to say nothing more.

The relations of things, as we have them, will remain the same while HE is pleased to continue the present constitution and arrangement of things. *Unalterable*, then, these relations must be; but, being the opposing of "his good pleasure," they cannot be *necessary;* for then, as already shown, they must be regarded as interwoven with his own existence: i. e., existing as the necessary relations of his being, which is itself necessary.*

To suppose these relations, or even the most abstract truths respecting them, to be *necessary*, would be to make them not the *relations* of things or beings, (or ultimately of the ONE SELF-EXISTENT BEING,) but *existences or things themselves.* If they existed *necessarily* and, of course, previous to a creation, *where* and *how* did they exist; unless in the discernment and prescience of the DIVINE MIND? The admission that they could exist *only there*, will itself be the full admission of all that has been asserted. *Truth*, *beauty* and *goodness*, then, are but the outflowings of his adorable perfection—of his infinite excellence; and their "*eternal*" laws are but transcripts of the same. Because of that perfection and excellence HE is gloriously above all control; and the origin and rule of all that is true and right, exists neither *above* nor *beside* HIM, but is found *in* HIM.

In his self-existence, therefore, as it "was, and is, and is to come," is to be found THE *one, the absolutely necessary truth:* all others are contingent, just so far as HE has made them so. Herein, is to be found, moreover, *the great, the final hypothesis*, upon which rests *the structure of the universe;* and which, too, undergirds and sustains that universe, in *all its relations.*

ART. XXIX.—*Results additional to those offered by Dr. Locke from his Three Experiments,* "*On Single and Double Vision produced by viewing objects with both eyes;*" by S. PEARL LATHROP, M.D.

FEELING an interest in the various branches of optics, I read with much pleasure, the article "On Single and Double Vision, &c.," by Dr. John Locke, in the January number of this Journal. Having acquired, as he says of himself, the power of voluntary convergence of the optical axes to an *extreme* degree without the aid of viewing near objects, I have verified the several experiments mentioned by him.

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* The seeming contradictions, *if any*, which this might involve in the case of truths called axiomatic, must be regarded as but *seeming;* and as arising from our inadequate comprehension of the relations in question, and inexplicable for reasons which may, perhaps, be similar to that which prevents us from discerning *how*, in eternity past, time e'er began; though we cannot escape from the belief of *the fact*, that it *somehow* occurred.